Connecting Polygonizations via Stretches and Twangs: Abstract

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Abstract

We show that the space of polygonizations of a fixed planar point set S of n points is connected by $O(n^2)$ "moves" between simple polygons. Each move is composed of a sequence of atomic moves called "stretches" and "twangs". These atomic moves walk between weakly simple "polygonal wraps" of S. These moves show promise to serve as a basis for generating random polygons.[1]

1 Introduction

This paper studies polygonizations of a fixed planar point set S of n points. Let the n points be labeled p_i, $i = 0, 1, \ldots, n-1$. A *polygonization* of S is a permutation σ of $\{0, 1, \ldots, n-1\}$ that determines a polygon: $P = P_\sigma = (p_{\sigma(0)}, \ldots, p_{\sigma(n-1)})$ is a simple (non-self-intersecting) polygon. We will abbreviate "simple polygon" to *polygon* throughout. As long as S does not lie in one line, which we will henceforth assume, there is at least one polygon whose vertex set is S. A point set S may have as few as 1 polygonization, if S is in convex position, and as many as $2^{\Theta(n)}$ polygonizations; see Fig. 1a.

Our goal in this work is to develop a computationally natural and efficient method to explore all polygonizations of a fixed set S. One motivation is the generation of "random polygons" by first generating a random S and then selecting a random polygonization of S. Generating random polygons efficiently is a long unsolved problem; only heuristics [AH96] or algorithms for special cases [ZSSM96] are known. Our work can be viewed as following a suggestion in the latter paper: "start with a ... simple polygon and apply some simplicity-preserving, reversible operations ... with the property that any simple polygon is reachable by a sequence of operations"

Our two operations are called *stretch* and *twang* (defined in Sec. 2). Neither is simplicity preserving, but they are nearly so in that they produce "polygonal wraps." A *polygonal wrap* $\mathcal{P}_\sigma$ is determined by a sequence σ of point indices that includes every index in $\{0, 1, \ldots, n-1\}$ at least once, such that there is a perturbation of the points in multiple contact that renders $\mathcal{P}_\sigma$ a simple closed curve through the perturbed points in σ order. Thus polygonal wraps disallow proper crossings but permit self-touching.

Fig. 1b shows a polygonal wrap with five double-contacts (p_1, p_4, p_5, p_8 and p_9).

Stretches and twangs take one polygonal wrap to another. A stretch followed by a natural sequence of twangs, which we call a *cascade*, constitutes a *forward move*. Forward moves take a polygon to a polygon, i.e., they are simplicity preserving. Reverse moves will not be described in this Abstract. A *move* is either a forward or a reverse move. We call a stretch or twang an *atomic move* to distinguish it from the more complex forward and reverse moves.

Our main result is that the configuration space of polygonizations for a fixed S is connected by forward/reverse moves, each of which is composed of a number of stretches and twangs, and that the diameter of the space is $O(n^2)$ moves. We can bound the worst-case number of atomic moves constituting a particular forward/reverse move by the geometry of the point set. Experimental results on random point sets show that, in the practical situation that is one of our motivations, the bound is small, perhaps even constant. We have also established loose bounds on the worst-case number of atomic operations as a function of n: an exponential upper bound and a quadratic lower bound. Tightening these bounds has so far proven elusive and is an open problem.

One can view our work as in the tradition of connecting discrete structures (e.g., triangulations, matchings) via local moves (e.g., edge flips, edge swaps). Our result is comparable to that in [vLS82], which shows connectivity of polygonizations in $O(n^3)$ edge-edge swap moves through intermediate self-crossing polygons. The main novelty of our work is that the moves, and even the stretches and twangs, never lead to proper crossings, for polygonal wraps have no such crossings.

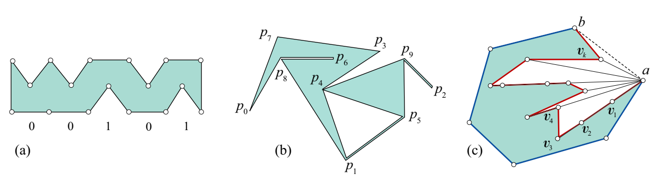

Figure 1: Examples. (a) A set of $n = 3k + 2$ points that admits 2^k polygonizations. (b) Polygonal wrap $\mathcal{P}_\sigma$ with $\sigma = (0, 8, 6, 8, 1, 5, 9, 2, 9, 4, 5, 1, 4, 3, 7)$ (c) A polygonization with one pocket with lid ab.

Let P be a polygonization of S. A hull edge ab that

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[1]See http://arxiv.org/abs/0709.1942 for the full version of this paper.

is not on ∂P is called a *pocket lid*, and the external polygon bounded by P and ab is a *pocket* of P. For a fixed hull edge ab, we define the canonical polygonization of S to be a polygon with a single pocket with lid ab (known to exist [CHUZ92]) in which the pocket vertices are ordered by angle about vertex a, and from closest to farthest from a if along the same line through a. We call this ordering the *canonical order* of the pocket vertices; see Fig. 1c.

2 Stretches and Twangs

In this Abstract, we do not have space to define the atomic moves STRETCH(e, v) and TWANG(abc), and instead rely on Figs. 2 and 3. Informally, if one views the polygon boundary as an elastic band, a stretch stretches e out to v, and a twang detaches the boundary from a vertex b and snaps it to b's convex side. The STRETCH(e, v) operation requires that v see a point x in the relative interior of e. The stretch is accomplished in two stages: (i) temporarily introduce two new "pseudovertices" on e in a small neighborhood of x (STRETCH$_0$ in Fig. 2), and (ii) remove the pseudovertices immediately using twangs.

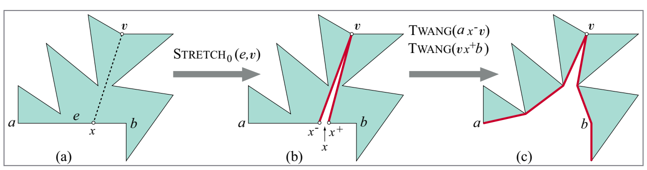

Figure 2: STRETCH(e, v) illustrated (a) v sees $x \in e$ (b) STRETCH$_0(e, v)$ (c) STRETCH(e, v).

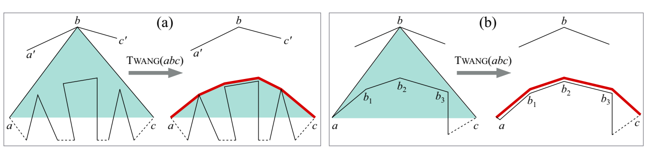

Figure 3: TWANG(abc) illustrated (a) TWANG(abc) replaces abc by sp(abc) (b) TWANG(abc) creates the hairpin vertex a and three doubled edges ab_1, b_1b_2 and b_2b_3.

That a twang cascade (see Fig. 4) eventually terminates is not immediate, but we establish both a geometric bound based on perimeter reduction, and combinatorial bounds of $\Omega(n^2)$ and $O(n^n)$ on the number of twangs in any twang cascade.

3 Algorithm & Connectivity

We first detail a SINGLE POCKET REDUCTION algorithm, which repeatedly picks a hull vertex v of some pocket A and attaches v to a pocket other than A. It terminates in $O(n)$ forward moves. Then the POCKET REDUCTION algorithm reduces multiple pockets to just

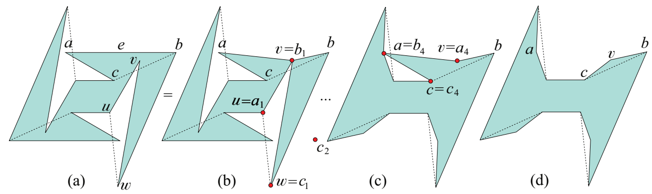

Figure 4: Forward move illustrated. (a) Initial polygon P; (b) After STRETCH(ab, v); (c) After TWANG$(a_1b_1c_1)$ and two more twangs not shown; (d) After TWANG$(a_4b_4c_4)$.

one, employing $\Theta(n^2)$ forward moves. Finally, the CANONICAL POLYGONIZATION algorithm converts the one pocket to the cononical form, again in $O(n)$ forward moves. Connectivity of the space of polygonizations follows by reducing two given polygonizations P_1 and P_2 to a common canonical form P_c, and then reversing the moves from P_c to P_2.

Theorem 1 *The space of polygonizations of a fixed n-point set is connected via a sequence of forward and reverse moves. Each node of the space has degree $\Omega(n)$, and the diameter of the space is $O(n^2)$ moves.*

This diameter bound is tight for our specific algorithm but might not be for other algorithms.

4 Open Problems

Our work leaves many interesting problems open. One unresolved question is narrowing the $\Omega(n^2)$ to $O(n^n)$ gap on the number of twangs in a cascade, thereby resolving the computational complexity of the polygon transformation algorithm. We would also like to establish a lower bound on the diameter.

It remains to be seen if the polygonization moves explored in this paper will be effective tools for generating random polygons. Finally, we are extending our work to 3D polyhedralizations of a fixed 3D point set.

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