Algorithms for Capacitated Rectangle Stabbing and Lot Sizing with Joint Set-Up Costs

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While capacity constraints appear naturally in many applications, the combinatorial and algorithmic nature of capacitated covering problems remains unresolved. Only a few capacitated problems were studied including the general case of Set Cover [9] and the restricted case of Vertex Cover [2, 7, 4]. In this work we consider a capacitated version of a covering problem, called rectangle stabbing. The geometric nature of the problem is used to obtain exact algorithms and constant approximation ratios for some versions of the problem.

The *rectangle stabbing* problem (RS) is a covering problem. Its uncapacitated version is defined as follows. The input is a finite set $\mathcal{U}$ of axis parallel rectangles and a finite set $\mathcal{S}$ of horizontal and vertical lines. A *cover* is a subset of $\mathcal{S}$ that intersects every rectangle in $\mathcal{U}$ at least once. The goal is to find a cover of minimum size. We denote the set of rectangles that a line $S \in \mathcal{S}$ intersects by $\mathcal{U}(S)$. Using this notation, an RS instance is simply a Set Cover instance in which the goal is to find a collection of subsets of the form $\mathcal{U}(S)$, the union of which equals $\mathcal{U}$. Without loss of generality, one may assume that the RS instance is discrete in the following sense [5]: rectangle corners have integral coordinates and lines intersect the axes at integral points.

In the one-dimensional version, the set $\mathcal{U}$ consists of horizontal intervals and the set $\mathcal{S}$ consists of points. This is the well known clique cover problem in interval graphs that is solvable in polynomial time [6]. The RS problem can be extended to d dimensions (d-RS). For $d \geq 3$, the set $\mathcal{U}$ consists of axis parallel d-dimensional rectangles (i.e., "boxes") and the set $\mathcal{S}$ consists of hyperplanes that are orthogonal to one of the d axes (i.e., "walls"). In the sequel we stick to the two-dimensional terminology, that is, we refer to $\mathcal{U}$ as a set of rectangles and to $\mathcal{S}$ as a set of lines.

Capacity constraints model the property that every covering object has limited resources that bound the number of elements it can cover. A limited covering ability of a covering object can occur in situations where covering each element consumes time or power. In the *capacitated d-dimensional rectangle stabbing* problem the input includes an integral capacity $c(S)$ for every line $S \in \mathcal{S}$. The capacity $c(S)$ bounds the number of rectangles that S can cover. This means that in the capacitated case one has to specify which line covers each rectangle. The assignment of rectangles to lines may not assign more than $c(S)$ rectangles to a line S. We discuss two variants of capacitated d-dimensional rectangle stabbing called covering with hard capacities (HARD-d-RS) and covering with soft capacities (SOFT-d-RS).

A cover in SOFT-d-RS is formally defined as follows. The input consists of a set $\mathcal{U}$ of d-dimensional axis-parallel rectangles and a set $\mathcal{S}$ of lines (i.e., hyperplanes) that are orthogonal to one of the d axis. Each line $S \in \mathcal{S}$ is given a nonnegative integral capacity $c(S)$. An *assignment* is a function $A : \mathcal{S} \to 2^{\mathcal{U}}$, where $A(S) \subseteq \mathcal{U}(S)$, for every S. A rectangle u is *covered* by a line S if $u \in A(S)$. An assignment A is a *cover* if every rectangle is covered by some line, i.e., $\bigcup_{S\in\mathcal{S}} A(S) = \mathcal{U}$. The *multiplicity* (or number of copies) of a line $S \in \mathcal{S}$ in an assignment A equals $\lceil |A(S)|/c(S) \rceil$. We denote the multiplicity of S in A by $\alpha(A, S)$. The *size* of a cover A is the sum $\sum_{S\in\mathcal{S}} \alpha(A, S)$. The goal is to find a cover of minimum size.

In HARD-d-RS, a line may appear at most once in a cover. Hence, in this case, a cover is an assignment A for which $|A(S)| \leq c(S)$, (or $\alpha(A, S) \in \{0, 1\}$) for every $S \in \mathcal{S}$. Note that SOFT-d-RS is a special case of HARD-d-RS, since given a SOFT-d-RS instance one can always transform it into a HARD-d-RS instance by duplicating each line $|\mathcal{U}|$ times.

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All the problems mentioned above have weighted versions, in which we are given a weight function w defined on the lines. In this case the weight of a cover A is $w(S) = \sum_S \alpha(A, S) \cdot w(S)$, and the goal is to find a cover of minimum weight.

We present a dynamic programming algorithm for weighted HARD-1-RS which implies also an exact algorithm for weighted SOFT-1-RS. The running time of the algorithm is $O((|\mathcal{U}|^2|\mathcal{S}|^2)(|\mathcal{U}| + |\mathcal{S}|))$. Our dynamic programming algorithm is motivated by a paper by Baptiste [1]. We note that a 2-approximation algorithm for weighted SOFT-1-RS whose running time is $O(|\mathcal{U}|^2 \cdot |\mathcal{S}|)$ was presented in [3] (see also [8]).

We present $3d$-approximation algorithm for SOFT-d-RS, where d is arbitrary. This algorithm solves an LP relaxation of the problem, and rounds it using the geometrical structure of the problem. For the case of hard capacities we show that the same technique can be used to obtain a bi-criteria algorithm for HARD-d-RS that computes solutions that are $4d$-approximate and use at most two copies of each line. We note that these techniques were extended in [8] to obtain an 8-approximation algorithm for HARD-1-RS. It follows that the integrality gap of the natural LP for HARD-1-RS is bounded by 8.

We present two hardness results. The first result mimics the hardness result given in [2], to show that weighted HARD-2-RS is Set Cover hard, even if all weights are in $\{0, 1\}$. The second hardness result proves that it is NP-hard to approximate d-RS with a ratio of $c \cdot \log d$, for some constant c. Note that the dimension d is considered here to be part of the input.

We also consider a variant of a *multi-item lot sizing inventory problem* (MILS) that generalizes HARD-1-RS. In this multi-item lot sizing problem there is a sequence of unit size *orders* (requests) denoted by $O_1, \ldots, O_n$ that need to be satisfied over a planning horizon of T discrete periods, indexed $t = 1, \ldots, T$. Each request O_i has a due date d_i, which means that it must be manufactured at some time period $s \leq d_i$. Production takes place in mixed batches of bounded capacity. Specifically, each time period $t = 1, \ldots, T$ is associated with a *capacity* $c(t)$ and weight $w(t)$. The capacity $c(t)$ bounds the number of requests that can be manufactured at time period t, and $w(t)$ is a fixed manufacturing cost at time period t.

In addition, there are costs for maintaining inventory, traditionally called *holding costs*. For example, if the request O_i is manufactured at some period $s \leq d_i$, there are holding costs incurred by carrying this request in inventory from period s to period d_i. Let $H_i(s)$ denote the holding cost incurred by request O_i given that it is manufactured at time period $s \leq d_i$. We assume that, for each $i = 1, \ldots, n$, the function $H_i(s)$ is non-negative and non-increasing in $s \in (0, d_i]$, i.e., shortening the holding time always results in lower holding cost. We also assume that the requests are indexed by increasing order of *importance*. Suppose that $i < j$. Then, $H_i(s_1) + H_j(s_2) \leq H_i(s_2) + H_j(s_1)$, for $s_1 < s_2 \leq \min\{d_i, d_j\}$. In words, shortening the holding time of the more important requests at the expense of extending the holding time of the less important requests never increases the overall holding costs. Assume that it is possible to satisfy all requests; namely, $\sum_{t=0}^{d_i} c(t) \geq i$, for $i = 1, \ldots, n$. The goal is to find a feasible manufacturing schedule with the least cost.

We show that our dynamic programming algorithm for weighted HARD-1-RS extends to solve also the multi-item lot sizing problem in time $O((n^2T^2)(n + T))$.

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