Lang's Universal Molecule Algorithm

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ABSTRACT

We present a Java implementation of Lang's Universal Molecule algorithm, alongside with a visualization of its interconnected structures: the input metric tree and compatible convex polygon, whose 2D crease pattern and 3D uniaxial base are computed by the algorithm. The Java applet, the video, as well as further references and accompanying materials are available on our web site[1]. We also include a recent example, found with the help of this implementation, of a Universal Molecule crease pattern which, as a flat-faced origami, is completely rigid; in particular, its corresponding uniaxial base cannot be reached through continuous folding without bending of the paper.

Categories and Subject Descriptors

I.3.5 [**Computational Geometry and Object Modeling**]: [Geometric algorithms, languages, and systems]

General Terms

Theory, Algorithms

Keywords

computational origami, Universal Molecule, TreeMaker

1. INTRODUCTION

In a beautiful paper from SoCG 1996 [4] (see also [5, 3]), Robert Lang introduced TreeMaker, an algorithm for designing origami crease patterns compatible with an input metric tree. His designs are distinguished by the existence of a flat-folded uniaxial base compatible with the input tree. At the core of TreeMaker lies the Universal Molecule algorithm, which is the focus of this video.

In origami design, line segments called *creases* are introduced on a sheet of paper to allow the paper to *fold* along them into a 3D state, called a *base*. The line segments form the *crease pattern*, which subdivides the paper into a set of polygonal *faces*. The TreeMaker algorithm creates creases that guarantee the existence of special 3D folded shapes, called *uniaxial bases*.

A base in which each face is perpendicular to a common plane is called *projectable*. Lang's bases project to a tree called the *shadow tree*. A *uniaxial base* is a Lang projectable base in which the boundary of the paper folds exactly onto its shadow tree, and the rest of the base lies above the projection plane. Lang's TreeMaker algorithm solves the following version of the origami design problem: given a metric tree and a sheet of paper (typically, a square or rectangle), produce a crease pattern which has a realization as a uniaxial base whose shadow tree is the input tree.

This video provides a visualization of Lang's Universal Molecule algorithm through a Java implementation, and is a companion to our recent papers [1, 2].

[1] `linkage.cs.umass.edu/origamiLang`

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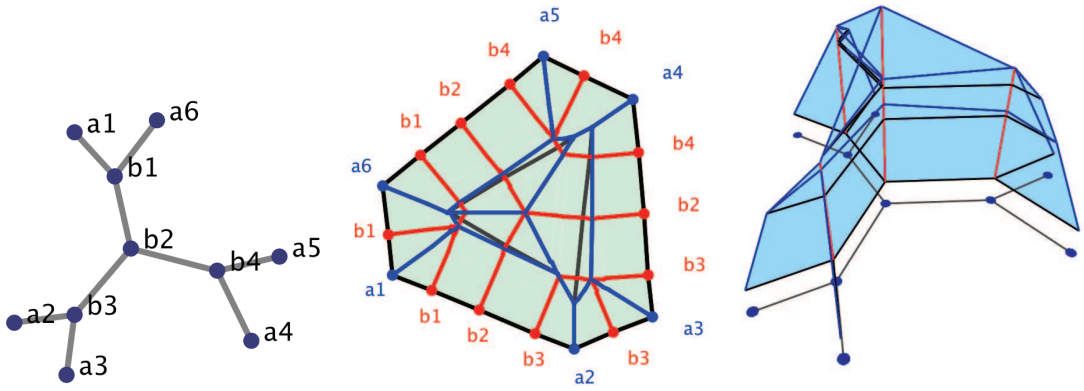
2. UNIVERSAL MOLECULE

Lang's Universal Molecule algorithm fills in the crease pattern for a (special type of) convex polygon, called a *Lang polygon*, as created by the initial, optimization step of TreeMaker. A Lang polygon is an example of an *origami molecule*, or a small polygonal subset of the paper which is folded somewhat independently of the rest of the paper. Lang's Universal Molecule algorithm is so called because it generates a crease pattern for any Lang polygon. The input is a topologically embedded metric tree and a Lang polygon for the tree. The output is a crease pattern for the polygon for which there exists a folded realization as a uniaxial base with shadow tree equal to (a geometric embedding of) the input tree.

A Lang polygon is a special case of a doubling cycle for the input tree. A *doubling cycle* is a polygon created by walking around the topologically embedded input tree. Each arc traversed along the walk corresponds to an edge of the same length in the doubling cycle. The leaf nodes are encountered only once and thus have unique corresponding vertices in the polygon, which we call *corners*. An interior node of degree d is encountered d times and is each time marked on the boundary. We call these vertices *marker* vertices. If a doubling cycle is convex and for each pair of corners the distance between the corners is greater than or equal to the distance in the tree between the corresponding leaf nodes, then the doubling cycle is a *Lang polygon*. The interior angle of any marker vertex is π.

The algorithm works by a *parallel sweep* process in which the edges of the polygon are moved inwards in a parallel fashion at constant speed. The interior sweeping polygon is called a *contour* and the distance the edges have moved at any point in the sweep process is called the *height* of the contour. The edges of the crease pattern are created by tracing the paths of the vertices (both corner and marker) of the polygon.



(a) Tree (b) Crease Pattern (c) Uniaxial Base

Figure 1: The applet illustrates the intercorrelation of the three structures that underlie the Universal Molecule algorithm.

As the contour shrinks inwards, an edge between two marker vertices maintains its length, while an edge between a marker and a corner shrinks. This corresponds to a shrinking of the tree by moving the leaves inward. At certain discrete heights one of two events occurs. A *contraction event* occurs when an edge between a corner and a marker vertex shrinks to a single point. All leaves incident to the internal node corresponding to the marker vertex shrink at once. This node is then a leaf node of the shrunken tree. A *branching event* occurs when the distance between two corner vertices becomes equal to the corresponding distance between the leaf nodes. If the sweeping process were to proceed past this point, the contour would cease to be a Lang polygon, since the distance between two leaf nodes would become less than the corresponding distance between the leaves. The contour is therefore split by adding an edge between the two corner nodes. Marker nodes are added so that the edge corresponds to the path in the tree between the two leaf nodes. This splits the contour into two contours, both of which are Lang polygons. The sub-tree for each of the contours is created by splitting the tree along the path between the two leaf nodes. The sweeping process then proceeds in parallel in each contour.

In 3D the sweeping contours correspond to a sweeping process of the shrinking tree upwards by the height of the contour. Conceptually, the sweeping process is a continuous process with discrete events. The algorithm, however, works by identifying the height of the next event and by recursing on the contour and sub-tree at that event. The base case is when the next event is a contraction of all edges to the same point.

Rigidity of Universal Molecules. Lang's Universal Molecule algorithm produces crease patterns for which uniaxial realizations exist. A question remains: can the uniaxial state be achieved by continuously folding the creased paper, in such a way that the faces are kept rigid and folds occur only along creases? In the video, we show an example found using our software implementation, which (as proven in [2]) answers this question in the negative. The example gives a totally inflexible crease pattern, which would not fold into its associated uniaxial base without further deformation (bending) of the paper.

3. SOFTWARE AND VIDEO

Our software is a Java Applet implementation in which the user defines the input tree and is then able to modify a doubling cycle for the tree to produce different Lang polygons. A slider allows the user to visualize the sweeping process. A 3D window shows the user a live visualization of the final folded base realizable with the computed crease-pattern.

After a brief overview of TreeMaker, the video discusses the intricacies of Lang's Universal Molecule algorithm, and shows a totally rigid example it may produce. The animations for the Universal Molecule segment were screen-captured from our implementation, with annotations added in post-processing with presentation software. The live demo of our implementation is available at our website `linkage.cs.umass.edu/origamiLang`.

4. ACKNOWLEDGEMENTS

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